

INFINITE MONOCHROMATIC SUMSETS FOR COLOURINGS OF THE REALS

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ABSTRACT. N. Hindman, I. Leader and D. Strauss proved that it is consistent that there is a finite colouring of \mathbb{R} so that no infinite sumset $X + X$ is monochromatic. Our aim in this paper is to prove a consistency result in the opposite direction: we show that, under certain set-theoretic assumptions, for any finite colouring c of \mathbb{R} there is an infinite $X \subseteq \mathbb{R}$ so that $c \upharpoonright X + X$ is constant.

1. INTRODUCTION

Neil Hindman’s famous sumset theorem states that for any finite colouring of the natural numbers \mathbb{N} , there is an infinite set X so that all sums of distinct elements of X are coloured the same. There is a striking difference however, if one allows repetitions in the sumsets; let us begin by recalling a famous open problem of J. Owings:

Problem 1.1 ([12]). *Is there a colouring of \mathbb{N} with 2 colours which is not constant on sets of the form $X + X = \{x + y : x, y \in X\}$ whenever $X \subseteq \mathbb{N}$ is infinite?*

The answer is yes if one is allowed to use 3 colours [6], but, surprisingly, Problem 1.1 is still unsolved. Hence, it is very natural to ask, what happens if instead of the natural numbers we colour the real numbers. The following theorem is a recent result of N. Hindman, I. Leader and D. Strauss:

Theorem 1.2 ([7]). *It is consistent (namely, it follows from $2^{\aleph_0} = |\mathbb{R}| < \aleph_\omega$) that there is a finite colouring c of \mathbb{R} such that c is not constant on any set of the form $X + X$ where $X \subseteq \mathbb{R}$ is infinite.*

The group $(\mathbb{R}, +)$ is isomorphic to the direct sum of continuum many copies of $(\mathbb{Q}, +)$ so we consider colourings of this direct sum. Thus, let $Q(\kappa)$ denote the group $\bigoplus_{\kappa} \mathbb{Q}$, i.e., the set of functions with finite support from κ to \mathbb{Q} with the operation of coordinatewise addition, and let the semigroup $N(\kappa) = \bigoplus_{\kappa} \mathbb{N}$ be defined similarly. As before, if X is a subset of one of the above (semi)groups, let $X + X = \{x + y : x, y \in X\}$, and note that repetitions are allowed in the sumsets.

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In order to prove Theorem 1.2, the authors of [7] showed the following: for each $n \in \omega$, there is a map $c : Q(\aleph_n) \rightarrow 2^{4+n} \cdot 9$ such that $c \restriction X + X$ is not constant for any infinite $X \subseteq Q(\aleph_n)$.

It was asked in [7, Question 2.9] if the conclusion of Theorem 1.2 is true in ZFC. Note that given an infinite semigroup $(G, +)$ and finite colouring $c : G \rightarrow r$ one can always find an infinite set $X \subseteq G$ so that both $\{x + y : x \neq y \in X\}$ and $\{x + x : x \in X\}$ are monochromatic. This is by the Ramsey theorem and the pigeon hole principle, respectively. However, the constant values of c on these two sets might differ. Arranging that these two colours are the same is the main difficulty in proving a positive partition result.

The purpose of this paper is to show that the conclusion of Theorem 1.2 can fail in some models of ZFC constructed with the use of a large cardinal.

Main Theorem. *Consistently relative to an ω_1 -Erdős cardinal, for any $c : N(2^{\aleph_0}) \rightarrow r$ with r finite there is an infinite $X \subseteq N(2^{\aleph_0})$ so that $c \restriction X + X$ is constant.*

The proof of the Main Theorem is a combination of various ideas from the six authors from between 2015 and 2017. I. Leader and P. A. Russell [11], and independently P. Komjáth¹ proved that if a colouring $c : N(\kappa) \rightarrow r$ is *canonical* in some sense² on a large set then infinite monochromatic sumsets can be found. Hence, the conclusion of the Main Theorem was known for $N(\kappa)$ instead of $N(2^{\aleph_0})$ where κ is large enough relative to the number of colours r . For example, $\kappa \geq \beth_\omega$ suffices for any finite r , since it allows the application of the Erdős-Rado partition theorem with high exponents; the interested reader can consult [11] or see Lemma 2.1 below.

Since 2^{\aleph_0} badly fails such strong positive partition relations, other ideas were required. D. Soukup and Z. Vidnyánszky analysed the situation further to see what alternative partition relation might suffice to carry out some form of such a canonization argument. It was S. Shelah who suggested using [13] and indeed, the last two authors found a way to combine [13, Theorem 3.1] with all the aforementioned machinery resulting in the Main Theorem.

Since $N(2^{\aleph_0})$ embeds into \mathbb{R} , we immediately have the desired conclusion.

Corollary. *Consistently relative to an ω_1 -Erdős cardinal, for any finite colouring c of \mathbb{R} there is an infinite $X \subseteq \mathbb{R}$ so that $c \restriction X + X$ is constant.*

Remark 1.3. We recently learned from Jing Zhang³ that he showed that the conclusion of our main theorem holds if one adds \aleph_ω many Cohen reals to a model of GCH.⁴ In particular, no large cardinals are necessary and the continuum can be as small as possible (cf. Theorem 1.2). We conclude our paper with further open problems.

Let us briefly mention the related problem of studying *uncountable* sumsets. Starting with [7] a sequence of papers [9, 14, 3, 4, 10, 2] elaborates on this question and its relatives; we refer the reader to the introduction of [4] for an excellent

¹Personal communication.

²That is, $c(x)$ only depends on the values of x in its 'support' but not the support itself.

³Personal communication.

⁴I.e., the Generalized Continuum Hypothesis which says that $2^\kappa = \kappa^+$ for any infinite cardinal κ .

summary. In particular, A. Rinot and D. Fernández-Bretón showed that one can colour \mathbb{R} with 2 colours (ω many colours) such that all colours appear on all sumsets of size \aleph_1 (sumsets of size 2^{\aleph_0} , respectively); both of these results are provable in ZFC [4] and hold even if repetitions are not allowed in the sumsets. This implies that our Main Theorem is optimal in the sense that we cannot hope to find uncountable monochromatic sumsets.

Throughout the paper, we will use standard notations and facts which can be found in [8]. Our proof is elementary; it only uses independence results as black boxes, so it can be understood without any familiarity with the techniques of forcing, large cardinals etc.

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2. THE PROOF OF THE MAIN THEOREM

The rest of our paper is devoted to proving our Main Theorem. We start by recalling the main ideas of [11] in Section 2.1 which will motivate both the somewhat technical canonization result of Shelah that we state in Section 2.2 and our proof of the Main Theorem in Section 2.3

2.1. Patterns and homogeneous sets. Let κ be a cardinal. Given $s \in \mathbb{N}^{<\omega}$ and $a \in [\kappa]^{|s|}$, we define $x = s * a \in N(\kappa)$ by $\text{supp}(x) = a$ and $x(a(i)) = s(i)$ where $\{a(i) : i < |s|\}$ is the increasing enumeration of a .

Now, if $c : N(\kappa) \rightarrow r$ is a colouring and $s \in \mathbb{N}^{<\omega}$, we define $c_s : [\kappa]^{|s|} \rightarrow r$ by

$$c_s(a) = c(s * a).$$

That is, any colouring c of the direct sum $N(\kappa)$ and each pattern s naturally defines a colouring c_s of finite subsets of the continuum. The main idea from [11] is the following: given an r -colouring c of $N(\kappa)$, there are $r + 1$ many patterns $(s_l)_{l \leq r}$ so that if each colouring c_{s_l} is constant on a large set $W \subseteq \kappa$ then we can find an infinite $X \subseteq N(\kappa)$ so that $c \upharpoonright X + X$ is constant.

Let us define these patterns: we let $s_l = (2, 2, \dots, 2, 2, 4, \dots, 4)$ where $2l$ -many 2 are followed by $(r - l)$ -many 4 for each $l \leq r$; see Figure 1. So, for a fixed r , we defined $(r + 1)$ -many patterns.

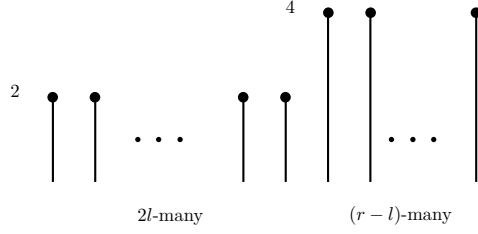


FIGURE 1. The pattern s_l .

Let us consider the special case of $r = 2$ when $s_0 = (4, 4)$, $s_1 = (2, 2, 4)$ and $s_2 = (2, 2, 2, 2)$. Suppose that $W \subseteq \kappa$ has order type $\omega + 1$ and all three colourings c_{s_0} , c_{s_1} and c_{s_2} are constant on W . Now, two of these constant values must agree, say both c_{s_0} and c_{s_1} are constant 0. Pick ordinals $\alpha_0 < \alpha_1 < \dots < \alpha_\omega$ in W and let $x_i = \{(\alpha_i, 2), (\alpha_\omega, 2)\}$ for $i < \omega$. It is easy to see that

$$c(2x_i) = c_{s_0}(\alpha_i, \alpha_\omega) = 0 = c_{s_1}(\alpha_i, \alpha_j, \alpha_\omega) = c(x_i + x_j)$$

for any $i < j < \omega$. So $c \upharpoonright X + X = 0$ for $X = \{x_i : i < \omega\}$.

In general, the following holds:

Lemma 2.1. *Suppose that $c : N(\kappa) \rightarrow r$ and there is some $W \subseteq \kappa$ of order type $\omega + r - 1$ so that each $c_{s_l} \upharpoonright [W]^{r+l}$ is constant. Then there is an infinite $X \subseteq N(\kappa)$ so that $c \upharpoonright X + X$ is constant.*

Proof. Suppose that $W = \{\alpha_i : i < \omega + r - 1\}$ satisfies the above requirements. Now, pick $l < k \leq r$ so that c_{s_l} and c_{s_k} have the same constant value on W . We define $x_i \in N(\kappa)$ for $i < \omega$ as follows: the support of x_i has the form $a \cup b_i \cup c$ where $a = \{\alpha_0, \alpha_1, \dots, \alpha_{2l-1}\}$, $c = \{\alpha_\omega, \dots, \alpha_{\omega+r-k-1}\}$ and the sets $\{b_i : i < \omega\}$ are pairwise disjoint of size $k - l$ selected from $\{\alpha_n : 2l \leq n < \omega\}$. We define $x_i \upharpoonright a$ constant 1 and $x_i \upharpoonright (b_i \cup c)$ constant 2. It is easy to check that

$$c(2x_i) = c_{s_l}(a \cup b_i \cup c) = c_{s_k}(a \cup b_i \cup b_j \cup c) = c(x_i + x_j)$$

for any $i < j < \omega$. So $c \upharpoonright X + X$ is constant for $X = \{x_i : i < \omega\}$. \square

Almost the same result as Lemma 2.1 appears in the proof of [11, Theorem 2] however the authors require a slightly larger order-type for W . We also learned from J. Zhang that he used the $r = 2$ case of Lemma 2.1 to note that for any 2-colouring of \mathbb{R} , one can find an infinite X so that $X + X$ is monochromatic.

However, it is well known that for $\kappa = 2^{\aleph_0}$ we cannot necessarily find the W that is required to apply the above result for $r > 2$. Still, we can (consistently) find a large subset W and a map $F : W \rightarrow 2^\omega$ such that the colourings c_{s_l} are simple on W in the sense that the colour of a tuple is essentially determined by the equivalence class of its F image with respect to a natural equivalence relation on finite tuples on 2^ω . Making the latter precise is the content of the next section.

2.2. Shelah's canonization theorem. Our goal now is to explain [13, Theorem 3.1 (2)] and we aimed to follow the definitions there. As usual, for distinct $s, t \in 2^\omega$ let $\Delta(s, t)$ stand for $\min\{n : s(n) \neq t(n)\}$.

We now define the equivalence relation that we mentioned earlier.

Definition 2.2. We say that $\bar{t}, \bar{s} \in (2^\omega)^k$ are *similar* iff for all $l_1, l_2, l_3, l_4 < k$:

- (1) $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$ iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
(2)

$$\bar{t}(l_3) \upharpoonright n <_{\text{lex}} \bar{t}(l_4) \upharpoonright n \text{ for } n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$$

iff

$$\bar{s}(l_3) \upharpoonright m <_{\text{lex}} \bar{s}(l_4) \upharpoonright m \text{ for } m = \Delta(\bar{s}(l_1), \bar{s}(l_2)),$$

(3)

$$\bar{t}(l_3)(n) = 0 \text{ for } n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$$

iff

$$\bar{s}(l_3)(m) = 0 \text{ for } m = \Delta(\bar{s}(l_1), \bar{s}(l_2)).$$

The above definition says that the ordered tuples \bar{t} and \bar{s} have the same branching pattern (conditions (1) and (2)) and the same values at the corresponding levels (condition (3)). Observe that there are only finitely many equivalence classes for similarity in $(2^\omega)^k$.

Now, given $d : [W]^k \rightarrow r$ and $F : W \rightarrow 2^\omega$, we say that

- (a) two tuples $\bar{\alpha} = \{\alpha_0 < \dots < \alpha_{k-1}\}, \bar{\beta} = \{\beta_0 < \dots < \beta_{k-1}\} \subset W$ are *F-similar* if $(F(\alpha_l))_{l < k}$ and $(F(\beta_l))_{l < k}$ are similar, and
(b) d is *F-canonical* if $d(\bar{\alpha}) = d(\bar{\beta})$ for any two *F-similar* tuples $\bar{\alpha}, \bar{\beta}$ from W .

We are ready to state Shelah's theorem.

Theorem 2.3. [13, Theorem 3.1 (2)] *Suppose that λ is an ω_1 -Erdős cardinal in V . Then there is a forcing notion \mathbb{P} so that $V^\mathbb{P}$ satisfies the following:*

- (1) $2^{\aleph_0} = \lambda$,
(2) $MA_{\aleph_1}(\text{Knaster})$ (see below), and
(3) for any $d : [\lambda]^k \rightarrow r$ (with $r, k \in \omega$) there is $W \in [\lambda]^{\aleph_1}$ and injective $F : W \rightarrow 2^\omega$ so that $d \upharpoonright [W]^k$ is *F-canonical*.

In particular, the above theorem says that no matter how large the number of colours r is, the restricted colouring $d \upharpoonright [W]^k$ assumes at most as many colours as the number of equivalence classes for similarity. For $k = 2$, this gives at most two colours.

We also mention that the forcing \mathbb{P} is of the form $\mathbb{P}_0 * \mathbb{P}_1$ where \mathbb{P}_0 is $< \lambda$ -closed of size λ and $\mathbb{P}_1 \in V^{\mathbb{P}_0}$ is ccc. \mathbb{P} collapses no cardinals $\leq \lambda$ so the continuum will be very large (i.e., a former large cardinal). In fact, we do not know if Theorem 2.3 could hold for say $2^{\aleph_0} = \aleph_{\omega+1}$ or if the use of large cardinals can be avoided in this result.

Let us omit the definition of $MA_{\aleph_1}(\text{Knaster})$ as we only need a particular corollary of this axiom.⁵

Theorem 2.4. *Suppose $MA_{\aleph_1}(\text{Knaster})$. Then for any $r < \omega$ and $g : \omega \times \omega_1 \rightarrow r$ there are $A \in [\omega]^\omega$, $B \in [\omega_1]^{\omega_1}$ so that $g \upharpoonright A \times B$ is constant.*

The conclusion above is abbreviated as $\left(\frac{\omega_1}{\omega}\right) \rightarrow \left(\frac{\omega_1}{\omega}\right)_r^{1,1}$. For the interested reader, we mention that $MA_{\aleph_1}(\text{Knaster})$ implies $\aleph_1 < \mathfrak{s}$ by [1, Theorem 7.7] and the discussion there, and $\aleph_1 < \mathfrak{s}$ implies $\left(\frac{\omega_1}{\omega}\right) \rightarrow \left(\frac{\omega_1}{\omega}\right)_r^{1,1}$ by [5, Claim 2.4]. The only important thing for us is that $\left(\frac{\omega_1}{\omega}\right) \rightarrow \left(\frac{\omega_1}{\omega}\right)_r^{1,1}$ holds in our model.

⁵The Knaster property and Martin's axiom are covered by [8] in detail.

2.3. Proving the Main Theorem. We prove that the conclusion of our Main Theorem holds in any model that satisfies conditions (1)-(3) of Theorem 2.3. So, for the rest of the paper, let us fix a colouring $c : N(2^{\aleph_0}) \rightarrow r$ with some $r < \omega$.

We use the patterns s_0, \dots, s_r as defined in Section 2.1. Note that the $r+1$ many colourings $c_{s_l} : [2^{\aleph_0}]^{r+l} \rightarrow r$ can be coded by a single map $d : [2^{\aleph_0}]^{2^r} \rightarrow r^r$. In turn, by applying Theorem 2.3, there is some $W \in [2^{\aleph_0}]^{\aleph_1}$ and injective $F : W \rightarrow 2^\omega$ so that $c_{s_l} \upharpoonright [W]^{r+l}$ is F -canonical for all $l \leq r$.

Our goal now is to define infinite subsets $A_0, \dots, A_{r-1} \subseteq W$ and a family of $r + \ell$ -tuples from elements of $\bigcup_{i < r} A_i$ for each $\ell \leq r$ (which we will call *canonical ℓ -candidates*) so that any two of them are F -similar. Steps 1-4 below detail the definition of canonical ℓ -candidates and the construction of the sets A_l which involve controlling the branching pattern and values at branching levels. This will allow us in Step 5 to carry out a similar argument to the proof of Lemma 2.1 and find an infinite X with $c \upharpoonright X + X$ constant.

Step 1. We start by selecting $A_0, \dots, A_{r-1} \subset W$ so that

- (1) $F''A_l < F''A_k$ (in 2^ω) for all $l < k < r$,
- (2) $|A_l| = \aleph_1$ for $l < r$, and
- (3) there is an increasing sequence $\nu_l < \omega$ (for $l < r - 1$) so that
 - $\Delta_F(\alpha, \alpha') = \nu_l$ for all $\alpha \in A_l, \alpha' \in A_k$ with $l < k < r$, and
 - $\Delta_F(\alpha, \alpha') > \nu_{r-2}$ for all $\alpha, \alpha' \in A_l$ with $l < r$.

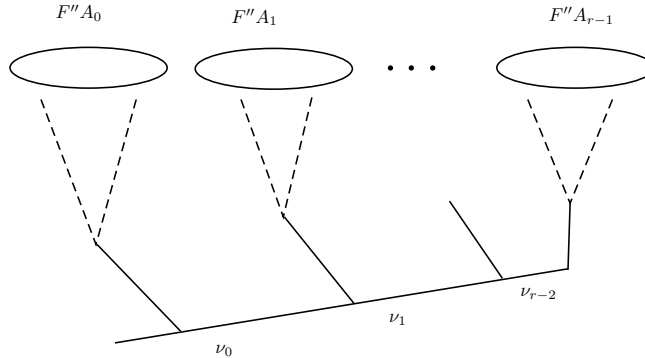


FIGURE 2. The r uncountable blocks from W with nicely ordered images on 2^ω .

See Figure 2. Now, this ensures that all tuples $(\alpha_0, \dots, \alpha_{r-1})_{<} \in A_0 \times \dots \times A_{r-1}$ belong to the same similarity type. Here, we use the notation $\{\alpha_i\}_{<}$ or $(\alpha_i)_{<}$ to denote $\alpha_0 < \alpha_1 < \dots$, in other words, the fact that the sequence of α_i 's is increasing in the ordering of ordinals.

Definition 2.5. An ℓ -candidate with respect to $\mathbf{A} = (A_l)_{l < r}$ is an $(r + \ell)$ -tuple of elements of W where we take two elements from each of $A_0, \dots, A_{\ell-1}$ and a single point from each A_k for $\ell \leq k < r$.

In particular, a 0-candidate selects a single element from each A_l . Observe that the choice of the A_l 's ensures that all 0-candidates are F -similar and so all elements $s_0 * \bar{\alpha}$ are assigned the same colour by c .

Step 2. In general, not all ℓ -candidates are necessarily similar. In particular, we need to look at the new splitting levels δ (coming from the first ℓ -many pairs) and make sure that the same values appear when we evaluate other branches at δ . We need to evaluate at branches lying above and below where the splitting occurs. We deal with the former first and the latter in Claim 2.9.

Claim 2.6. *There is $A'_l = \{\alpha'_i : i \leq \omega\} < \subseteq A_l$ for each $l < r$ so that*

- (1) *$F \upharpoonright \bigcup_{l < r} A'_l$ is order preserving and $\sup_{i \in \omega} F(\alpha'_i) = F(\alpha'_\omega)$ in 2^ω ,*
- (2) *there are strictly increasing $(\delta'_i)_{i < \omega}$ so that*
 - (a) *$\Delta_F(\alpha'_i, \alpha'_j) = \delta'_i$ for all $i < j \leq \omega$, and*
 - (b) *$\delta'_i < \delta'_k < \delta'_{i+1}$ for $l < k < r$ and $i < \omega$,*
- (3) *there are $u_l : \{l+1, \dots, r-1\} \rightarrow 2$ for $l < r$ such that $u_l(k) = F(\alpha)(\delta'_i)$ for any $\alpha \in A_k$ and $l < k < r$.*

See Figure 3 for a picture; we have marked the important splitting levels on later branches, with the same symbols marking the same values.

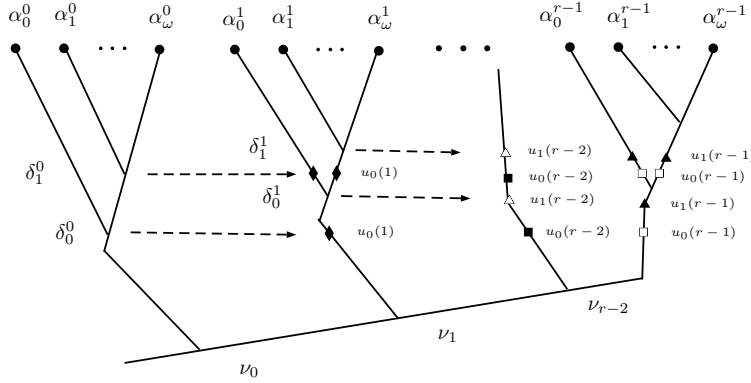


FIGURE 3. Selecting subsequences with the important positions marked.

Proof. First, shrink A_0 to some A''_0 of type $\omega + 1$ so that $F \upharpoonright A''_0$ is increasing, continuous, and (2)(a) is satisfied; next, replace A_1, \dots, A_{r-1} by their uncountable subsets so that every element of their union is greater than every element of A''_0 (we use the notation A_1, \dots, A_{r-1} for the shrunk sets as well). Note that (2)(a) will still hold no matter how we shrink A''_0 further (just the splitting sequence is redefined).

Define

$$g_{0k} : (A''_0 \setminus \{\max A''_0\}) \times A_k \rightarrow 2$$

for $0 < k < r$ by $g_{0k}(\alpha'_i, \beta) = F(\beta)(\delta'_i) \in 2$. Now, apply $(\omega_1) \rightarrow (\omega_1)_2^{1,1}$ to each $g_{01}, g_{02}, \dots, g_{0r-1}$, successively shrinking $A''_0 \setminus \{\max A''_0\}$ $(r-1)$ -many times and each A_k once. The constant values $u_0(k)$ for g_k will define $u_0 : r \setminus 1 \rightarrow 2$. Note that we made sure that (3) is satisfied no matter how we shrink A''_0 or A_l for $0 < l < r-1$ in later steps.

We move onto A_1 and proceed similarly: first, shrink A_1 to a type $\omega + 1$ sequence so that the restriction of F is increasing, continuous, and (2)(a) is satisfied. Then shrink the sets A_2, \dots, A_{r-1} so that every element of their union is greater than

the elements of A_1 . Next, shrink each set further so that (3) holds for $l = 1$ using $(\omega_1) \rightarrow (\omega_1)_2^{1,1}$; this defines u_1 . Then we move to A_2 etc.

Notice that this process yields $A'_l \subseteq A_l$ so that (1), (2)(a) and (3) holds. Finally, we do a final (simultaneous) shrinking to ensure (2)(b). \square

Again, to make notation lighter, we drop the primes i.e., we forget about the original uncountable A_l defined in Step 2, and call our new $\omega + 1$ -sequences from Claim 2.6 A_l (instead of A'_l).

Now, what decides the similarity type of an ℓ -candidate with respect to this new sequence $\mathbf{A} = (A_l)_{l < r}$? Firstly, we will focus only on ℓ -candidates of a certain specific form:

Definition 2.7. A *canonical ℓ -candidate* is an ℓ -candidate with respect to \mathbf{A} of the form

$$\bar{\alpha} = (\alpha_{i_0}^0, \alpha_{j_0}^0, \dots, \alpha_{i_{\ell-1}}^{\ell-1}, \alpha_{j_{\ell-1}}^{\ell-1}, \alpha_{i_\ell}^\ell, \dots, \alpha_{i_{r-1}}^{r-1})$$

so that

- (1) $i_k < j_k \leq \omega$ for $k < \ell$ and $i_0 \leq i_1 \leq \dots \leq i_{r-1} \leq \omega$, and
- (2) if $j_k \neq \omega$ then $\delta_{j_k}^k > \delta_{i_{\ell-1}}^{\ell-1}$.

For example, if we fix $\ell < k < r$ and $i < \omega$ then

$$\{\alpha_0^l, \alpha_\omega^l : l < \ell\} \cup \{\alpha_i^l : \ell \leq l < k\} \cup \{\alpha_\omega^l : k \leq l < r\}$$

is a canonical ℓ -candidate. The corresponding indices are $i_l = 0 < j_l = \omega$ for $l < \ell$, $i_l = i$ for $\ell \leq l < k$ and $i_k = \omega$ for $k \leq l < r$. Condition (2) is vacuously satisfied. These particular canonical ℓ -candidates will have an important role later.

Step 3. Now, note that any 0-candidate is a canonical 0-candidate and they are all pairwise F -similar. Our next claim is that for a fixed sequence of i 's the F -similarity type of a canonical ℓ -candidate does not depend on the choice of the j 's.

Claim 2.8. Let $\mathbf{i} = \{i_0 \leq i_1 \leq \dots \leq i_{r-1}\}$ be a sequence of ordinals $\leq \omega$ and $\bar{\alpha} = (\alpha_{i_0}^0, \alpha_{j_0}^0, \dots, \alpha_{i_{\ell-1}}^{\ell-1}, \alpha_{j_{\ell-1}}^{\ell-1}, \alpha_{i_\ell}^\ell, \dots, \alpha_{i_{r-1}}^{r-1})$ and $\bar{\alpha}' = (\alpha_{i_0}^0, \alpha_{j'_0}^0, \dots, \alpha_{i_{\ell-1}}^{\ell-1}, \alpha_{j'_{\ell-1}}^{\ell-1}, \alpha_{i_\ell}^\ell, \dots, \alpha_{i_{r-1}}^{r-1})$ be two canonical ℓ -candidates (for some sequences $(j_l)_{l < \ell}$ and $(j'_l)_{l < \ell}$). Then $\bar{\alpha}$ and $\bar{\alpha}'$ are F -similar.

Proof. Observe first that for a canonical ℓ -candidate $\bar{\alpha}$, the similarity type of $F(\bar{\alpha})$ is decided by the values of the $2k$ -sequences

$$(F(\alpha_{i_0}^0)(\delta_{i_k}^k), F(\alpha_{j_0}^0)(\delta_{i_k}^k), \dots, F(\alpha_{i_{k-1}}^{k-1})(\delta_{i_k}^k), F(\alpha_{j_{k-1}}^{k-1})(\delta_{i_k}^k))$$

for each $k < \ell$ (see Figure 4): indeed, by the definition of similarity we only need to consider the values of the reals in $F''\bar{\alpha}$ at splitting levels and (3) of Claim 2.6 guarantees that for the splitting at $\delta_{i_k}^k$ we only need to check the values of reals below $F''A_k$. Also, as $\Delta_F(\alpha_{j_m}^m, \alpha_\omega^m) = \delta_{j_m}^m > \delta_{i_k}^k$ whenever $m < k$ and $\alpha_{j_m}^m \neq \alpha_\omega^m$, we get that

$$\begin{aligned} & (F(\alpha_{i_0}^0)(\delta_{i_k}^k), F(\alpha_\omega^0)(\delta_{i_k}^k), \dots, F(\alpha_{i_{k-1}}^{k-1})(\delta_{i_k}^k), F(\alpha_\omega^{k-1})(\delta_{i_k}^k)) = \\ & = (F(\alpha_{i_0}^0)(\delta_{i_k}^k), F(\alpha_{j_0}^0)(\delta_{i_k}^k), \dots, F(\alpha_{i_{k-1}}^{k-1})(\delta_{i_k}^k), F(\alpha_{j_{k-1}}^{k-1})(\delta_{i_k}^k)). \end{aligned}$$

The first part of the equation does not depend on the sequence j_0, \dots, j_{l-1} so the second part neither. This, and our first observation, finishes the proof of the claim. \square

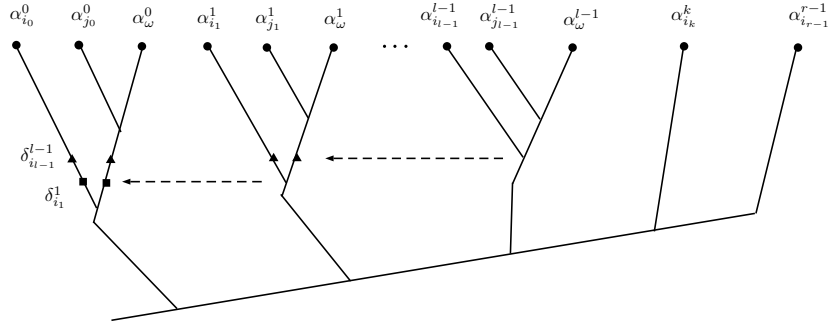


FIGURE 4. The important positions marked for deciding similarity types.

For a sequence $\mathbf{i} = \{i_0 \leq i_1 \leq \dots \leq i_{r-1}\}$, let us define $\mathfrak{c}(\mathbf{i}, k) : 2k \rightarrow 2$ by

$$\mathfrak{c}(\mathbf{i}, k)(2m) = F(\alpha_{i_m}^m)(\delta_{i_k}^k) \text{ and } \mathfrak{c}(\mathbf{i}, k)(2m+1) = F(\alpha_{\omega}^m)(\delta_{i_k}^k)$$

for $m < k$ and $i_k \neq \omega$; if $i_k = 0$, we declare $\mathfrak{c}(\mathbf{i}, k) = \emptyset$.

If we work with multiple \mathbf{A} sequences at the same time then we mark the above functions as $\mathfrak{c}_{\mathbf{A}}(\mathbf{i}, k)$. With this new notation, in Claim 2.8 we argued that the F -similarity type of a canonical ℓ -candidate with respect to \mathbf{A} is decided solely by the sequence $(\mathfrak{c}_{\mathbf{A}}(\mathbf{i}, k))_{k < r}$.

Step 4. Next, we thin out each A_l further so that the function $\mathfrak{c}(\cdot, k)$, at a fixed k , gives the same values for *any choice* of \mathbf{i} .

Claim 2.9. *There are $\tilde{A}_l = \{\tilde{\alpha}_i^l : i \leq \omega\}_< \subseteq A_l$ for $l < r$ so that $\tilde{\mathbf{A}} = (\tilde{A}_l)_{l < r}$ still satisfies the conditions of Claim 2.6 and there are $\mathfrak{c}(k) : 2k \rightarrow 2$ so that for any $\mathbf{i} = \{i_0 \leq i_1 \leq \dots \leq i_{r-1}\}$*

$$\mathfrak{c}_{\tilde{\mathbf{A}}}(\mathbf{i}, k) = \mathfrak{c}(k).$$

Proof. We can do this by a simple application of the classical (r -dimensional) Ramsey theorem. Note that we associated with any choice of $\mathbf{i} = \{i_0 \leq i_1 \leq \dots \leq i_{r-1}\} \in [\omega]^r$ a sequence $\psi(\mathbf{i}) = (\mathfrak{c}(\mathbf{i}, k))_{k < r}$. So, we defined

$$\psi : [\omega]^r \rightarrow \prod_{k < r} (2^{2k} \cup \{\emptyset\}),$$

and hence there is an $I \in [\omega]^\omega$ and $(\mathfrak{c}(k))_{k < r}$ so that $\psi \upharpoonright [I]^r$ is constant $(\mathfrak{c}(k))_{k < r}$.

Now, we simply let $\tilde{A}_l = \{\alpha_i^l : i \in I \cup \{\omega\}\}$ for $l < r$. It is easily checked that the conclusions of Claim 2.6 are still satisfied. \square

Again, we drop the tilde notation and assume that \mathbf{A} satisfies the conclusions of Claim 2.9.

Corollary 2.10. *For any $\ell \leq r$, any two canonical ℓ -candidates with respect to \mathbf{A} are F -similar.*

Step 5. We are ready to find our infinite monochromatic sumset. Recall that the colourings c_{s_ℓ} are F -canonical on $[\bigcup_{i < r} A_i]^{r+l}$ so the colour of a sequence is determined solely by its similarity type. In previous steps, we achieved that for a fixed $\ell \leq r$ all canonical ℓ -candidates are F -similar. In turn, as we have $r+1$ many patterns, we can find $\ell < k \leq r$ so that the map c_{s_ℓ} on canonical ℓ -candidates and the map c_{s_k} on canonical k -candidates assumes the same constant value.

Claim 2.11. *There is an $X = \{x_i : i < \omega\} \subseteq N(2^{\aleph_0})$ with $\text{supp}(x_i) \subset W$ so that*

- (1) *$\text{supp}(2x_i)$ is a canonical ℓ -candidate and $2x_i = s_\ell * \text{supp}(2x_i)$, and*
- (2) *$\text{supp}(x_i + x_j)$ is a canonical k -candidate and $x_i + x_j = s_k * \text{supp}(x_i + x_j)$*

for all $i < j < \omega$.

In particular, $c(2x_i) = c(x_i + x_j)$ so $c \upharpoonright X + X$ is constant as desired.

Proof. We let

$$\text{supp}(x_i) = \{\alpha_0^l, \alpha_\omega^l : l < \ell\} \cup \{\alpha_i^l : \ell \leq l < k\} \cup \{\alpha_\omega^l : k \leq l < r\}$$

and define x_i as follows:

$$x_i(\alpha) = \begin{cases} 1, & \text{for } \alpha \in \{\alpha_0^l, \alpha_\omega^l : l < \ell\}, \\ 2, & \text{for } \alpha \in \{\alpha_i^l : \ell \leq l < k\} \cup \{\alpha_\omega^l : k \leq l < r\}. \end{cases}$$

It is clear that $\text{supp}(2x_i) = \text{supp}(x_i)$ is a canonical ℓ -candidate with respect to \mathbf{A} . Moreover, $2x_i = s_\ell * \text{supp}(2x_i)$ by the definition of x_i .

Now

$$\text{supp}(x_i + x_j) = \{\alpha_0^l, \alpha_\omega^l : l < \ell\} \cup \{\alpha_i^l, \alpha_j^l : \ell \leq l < k\} \cup \{\alpha_\omega^l : k \leq l < r\}$$

for any $i < j < \omega$. This is a canonical k -candidate; indeed, we need to check (2) i.e. that $\delta_j^l > \delta_i^{k-1}$ for all $\ell \leq l < k$. This is clear from Claim 2.6 however. \square

3. OPEN PROBLEMS

There are various problems that remain open at this point. In order to state these questions concisely, we introduce the following notation: given some additive structure $(A, +)$, let $h(A)$ denote the minimal r so that there is an r -colouring of A with no monochromatic set of the form $X + X$ for some infinite $X \subseteq A$. Note that the larger $h(A)$ is, the stronger partition property A satisfies.

Now, with this new notation, we restate Owings' problem:

Problem 3.1. [12] *Does $h(\mathbb{N}) = 2$ hold?*

It was recently noted by J. Zhang⁶ that $h(\mathbb{R}) = 2$ holds in ZFC. However, the next problem remains open:

Problem 3.2. *Does $h(\mathbb{Q}) = 2$ hold?*

It is easy to see that $h(Q(\kappa)) \leq \aleph_0$ for any κ and so the conclusion of our main theorem can be rephrased as $h(\mathbb{R}) = \aleph_0$.

Problem 3.3. *Does $h(\mathbb{R}) = \aleph_0$ hold if 2^{\aleph_0} is real valued measurable?*

Also, recall that $h(Q(\aleph_n)) \leq 2^{4+n} \cdot 9$ [7]. Now, we ask the following:

⁶Personal communication.

Problem 3.4. *What is the asymptotic behaviour of the function $n \mapsto h(Q(\aleph_n))$? Is it truly exponential?*

A linear lower bound, under GCH, is provided by [11].

Probably the most intriguing question about large direct sums is the following:

Problem 3.5. *Is $h(Q(\aleph_\omega)) = \aleph_0$ provable in ZFC?*

GCH implies $h(Q(\aleph_\omega)) = \aleph_0$ [11] and a positive answer in ZFC would strengthen our main theorem.

The study of Abelian semigroups and finding larger, uncountable monochromatic sumsets has been already started in [4, 11]. It would also be interesting to see what one can say about unbalanced sumsets i.e., sets of the form $X + Y$ with $X, Y \subseteq A$. More precisely:

Problem 3.6. *Given an additive structure $(A, +)$ and r , characterize those (κ, λ) such that whenever $c : A \rightarrow r$ then there is $X, Y \subseteq A$ with $|X| = \kappa, |Y| = \lambda$ and $c \restriction X + Y$ constant.*

It would be interesting to analyze similar questions in non Abelian settings as well.

REFERENCES

1. Blass, Andreas. "Combinatorial cardinal characteristics of the continuum." Handbook of set theory. Vols. 1, 2, 3, 395–489, Springer, Dordrecht, 2010.
2. Carlucci, Lorenzo. "A note on Hindman-type theorems for uncountable cardinals" arXiv:1703.06706
3. Fernández-Bretón, David. "Hindman's Theorem is only a Countable Phenomenon." Order (2015): 1-9.
4. Fernández-Bretón, David and Assaf Rinot, "Strong failures of higher analogs of Hindman's Theorem", to appear in Trans. Amer. Math. Soc. (Published electronically: May 31, 2017)
5. Garti, Shimon, and Saharon Shelah. "Strong polarized relations for the continuum." Annals of Combinatorics 16.2 (2012): 271-276.
6. Hindman, Neil. "Partitions and sums of integers with repetition." Journal of Combinatorial Theory, Series A 27.1 (1979): 19-32.
7. Hindman, Neil, Imre Leader, and Dona Strauss. "Pairwise sums in colourings of the reals." Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg. Springer Berlin Heidelberg, 2015.
8. Jech, Thomas. Set Theory, The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.
9. Komjáth, Péter. "A certain 2-coloring of the reals." Real Anal. Exchange 41 (2016), no. 1, 227–231.
10. Komjáth, Péter. "A Ramsey statement for infinite groups.", to appear in Combinatorica.
11. Leader, Imre and Paul A. Russell. "Monochromatic infinite sumsets", preprint, ArXiv: 1707.08071.
12. J. Owings, Problem E2494, Amer. Math. Monthly 81 (1974), 902.
13. Shelah, Saharon. "Strong partition relations below the power set: consistency. Was Sierpinski right? II,[Sh 288]." Coll. Math. Soc. J. Bolyai 60 (1991): 1-32.
14. Soukup, Dániel T. and William Weiss. "Sums and anti-Ramsey colorings of \mathbb{R} ", personal notes (http://www.logic.univie.ac.at/~soukupd73/finset_coloring.pdf)

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